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New perspective in the theory of second-order stochastic processes

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Abstract. Starting from Langevin equations, we derive Fokker–Planck-like equations (FPLEs) for the joint distribution of displacements and velocities, $p(x, v, t)$, for a particle in a Gaussian random force field, firstly for the inertial process (i.e. in the absence of a frictional force) with a time correlated force, and secondly, for the Brownian motion with a white-noise force. From two different forms of the Langevin equation as coupled or decoupled first-order equations, we obtain two different forms of FPLEs for each one of these processes. In the inertial case one of the FPLEs reduces to an equation studied earlier by the author, while the other coincides with the equation obtained recently by Drory from an involved time discretization. In the Brownian motion case one of the FPLEs coincides with the free-particle Kramers equation obtained from the Fokker–Planck formalism for Markov processes. For each one of these processes the exactly determined initial value solutions of the two FPLEs are found to coincide. It follows, in particular, that the Markovian character of $p(x, v, t)$ for the Brownian motion is respected, regardless of which FPLE is used for defining it. Furthermore, for each process the two FPLEs lead to the same diffusion-like equation for the marginal distribution of displacements. The latter have been used elsewhere for studying first passage times, as well as survival probabilities in the presence of traps.

1. Introduction

The most common example of a second-order stochastic process is provided by the Langevin equation for a Brownian particle,

$$\ddot{x} = -\gamma\dot{x} + \frac{1}{m}f(t) \tag{1}$$

where $m\gamma$ is the friction coefficient and $f(t)$ is a fluctuating Gaussian inertial force. In discussing Brownian motion we restrict ourselves in the following to the usual white-noise case, i.e.

$$\langle f(t)f(t')\rangle = f_0^2\delta(t-t') \qquad \langle f(t)\rangle = 0 \tag{2}$$

where the displacement $x(t) - \bar{x}$ from a deterministic mean position $\bar{x}$ is Markovian for large γ, as is well known.

The case where $\gamma = 0$ describes a different, non-Markovian second-order process,

$$\ddot{x} = \frac{1}{m}f(t) \tag{3}$$

which is referred to as the inertial process. In this case we shall assume $f(t)$ to be a correlated Gaussian noise of Ornstein–Uhlenbeck (OU) type:

$$\langle f(t)f(t')\rangle = f_0^2 h(t-t') \tag{4}$$

$$h(t-t') = (2\tau)^{-1}\exp(-|t-t'|/\tau) \tag{5}$$

which reduces to (2) for vanishing correlation time, $\tau \to 0$.

Among various applications of the process (3) we recall its recent use for modelling reaction–diffusion processes in an inertial force dominated regime [1]. More generally, however, there are many examples of second-order stochastic processes in physics, chemistry and engineering (e.g. control, filtering communications) of which the processes (1) and (3) may be regarded as prototypes.

Detailed theoretical studies of the inertial process with a finite τ have appeared in the last few years. In a paper [2], hereafter referred to as I, the author presented explicit results for the joint probability distribution, $p(x, v, t)$, of displacements and velocities ($v = \dot{x}$), and for the marginal distributions, $p(x, t)$, $p(v, t)$, for this process. Our detailed expressions for the distributions were restricted to the short ($t \ll \tau$)- and long time ($t \gg \tau$)-limits, respectively. Analogous studies have also been performed for dichotomous noise [3], and for arbitrary Gaussian noise, with emphasis on anomalous diffusion and fractal behaviour [4]. Finally, first passage times and survival probabilities in the presence of additional trapping centres have been discussed for the inertial diffusion process (1) with OU noise [5].

In an interesting recent paper [6], Drory presented a general formalism for higher-order stochastic processes, which he described explicitly in the context of second-order processes. His approach is based on discretizing such a process by breaking-up the continuous time variable into a finite number of steps. By applying his analysis to the inertial process (3) with a Gaussian OU noise (4)–(5), Drory obtained the exact joint distribution $p(x, v, t)$ for a particle which started from an arbitrary point x_0, with an arbitrary velocity v_0, at $t = 0$. Moreover, he showed that this distribution is the initial value solution, with $p(x, v, 0) = \delta(x - x_0)\delta(v - v_0)$, (also called the fundamental solution) of a new Fokker–Planck-like equation (FPLE)† which he derived from his discretization procedure, as well as of the quite different FPLE obtained in I. From this it follows that the FPLE derived in I and in [6] for the process (3)–(5) are equally acceptable for describing the temporal evolution of $p(x, v, t)$ from the initial distribution $\delta(x - x_0)\delta(v - v_0)$. This is no longer expected to be so in cases where boundary conditions are added to the initial value condition, as required, for example, when studying first passage times (and survival probabilities in the presence of traps) for the process (3) in the two-variable (x, v) space [7]. Indeed the solutions of the two FPLEs for this mixed initial value–boundary value problem [8] will generally be different [6]. However, this statement adapted from [6] must be taken with caution, since its verification, by obtaining boundary value solutions from a fundamental solution of the FPLE, is notoriously difficult.

In view of the widespread use of FPLEs in many different areas, the findings of Drory about their non-uniqueness for higher-order stochastic processes are at first surprising. In particular, because of the possible consequences of this fact for boundary value problems of current interest, it is important to clarify its origin and to examine whether the unpleasant non-uniqueness also affects FPLEs for marginal distributions or whether, on the contrary, it is restricted to the FPLE for the joint distribution of the random variables. For this purpose we reconsider the Gaussian inertial process in section 2 to show that the existence of the two distinct forms of the FPLE for $p(x, v, t)$ arise from two different ways of expressing

† In the following we use the term FPLEs for evolution equations for joint distributions of displacements and velocities (and for the corresponding marginal distributions) which are derived directly from Langevin dynamical equations for Gaussian processes, without further considerations about their Markovian or non-Markovian nature. In particular, evolution equations for Markov processes obtained in this way will also be referred to as FPLEs. Thus, in section 4, the familiar Fokker–Planck equation (FPE) for Brownian motion is obtained as a special form of FPLE.

the Langevin equation (3) in terms of the first-order equations for x and for $v = \dot{x}$. We also show that the two FPLEs reduce to a common diffusion-like equation for $p(x, t)$, which is related to the unique way of casting (3) in the form of a first-order process $x(t)$. This diffusion-like equation thus provides a reliable starting point for studying first passage times and survival probabilities for the inertial process [5].

Alternatively, one would like to know to what extent the detailed conclusions derived by Drory for the inertial process are similar for other processes and, in particular, for its close relative, the ubiquitous Brownian motion process, described by (1) and (2). The Brownian motion process is studied in section 3 by a procedure similar to that used in section 2. This analysis is of interest partly because the Brownian process is Markovian (at least for large γ), unlike the inertial process for which $x(t)$ is non-Markovian even for white noise. Therefore, one of the two FPLEs obtained for this case is expected to reduce to the familiar Fokker–Planck equation (FPE) for free Brownian motion, the so-called Kramers equation. As in the case of the inertial process, we present a complete discussion of initial value solutions of the two FPLEs and of the corresponding diffusion-like equations for the marginal distribution $p(x, t)$ for Brownian motion. In section 4 the results of the previous sections are analysed in the light of the non-Markovian and Markovian characters of the inertial and Brownian motion processes, respectively.

2. The inertial process

Following I, the probability density $p(x, v, t)$ of a displacement x and a velocity v of a particle described by (3) is given by the noise-averaged expression

$$p(x, v, t) = \langle \delta(x - x(t))\delta(v - v(t)) \rangle \tag{6}$$

where $x(t)$ and $v(t)$ are the solutions of the coupled stochastic equations

$$\dot{x}(t) = v(t) \tag{7a}$$

$$\dot{v}(t) = \frac{1}{m} f(t) \tag{7b}$$

which may be viewed as defining a velocity at each point of the two-dimensional (x, v) phase space [9]. In order to derive FPLEs for an arbitrary Gaussian noise we differentiate both sides of (6) with respect to time, obtaining (with $\delta(\boldsymbol{u}-\boldsymbol{u}(t)) \equiv \delta(x-x(t))\delta(v-v(t))$),

$$\frac{\partial p(x, v, t)}{\partial t} = -\frac{\partial}{\partial x}\left\langle \frac{\partial x(t)}{\partial t}\delta(\boldsymbol{u} - \boldsymbol{u}(t))\right\rangle - \frac{\partial}{\partial v}\left\langle \frac{\partial v(t)}{\partial t}\delta(\boldsymbol{u} - \boldsymbol{u}(t))\right\rangle. \tag{8}$$

We now observe that there are two distinct ways of proceeding from this equation. First, one may use for the phase-space velocities the original expressions (7) describing coupled first-order processes. This procedure, which was used in I, is analogous to that followed in the general derivation of FPEs for nth-order stochastic processes based on Van Kampen's lemma [9] and leads to [2]

$$\frac{\partial p(x, v, t)}{\partial t} = -v\frac{\partial p}{\partial x} - \frac{1}{m}\frac{\partial}{\partial v}\langle f(t)\delta(\boldsymbol{u} - \boldsymbol{u}(t))\rangle. \tag{9}$$

Another procedure consists in substituting in (8) the integral forms for the 'velocities' (with v_0 the initial velocity), in terms of the random force, namely

$$\dot{x}(t) = v_0 + \frac{1}{m}\int_0^t \mathrm{d}t'' f(t'') \tag{10a}$$

$$\dot{v}(t) = \frac{1}{m} f(t) \tag{10b}$$

which amounts to defining *distinct first-order processes* $x(t)$ and $v(t)$. In this case (8) takes the form

$$\frac{\partial p(x,v,t)}{\partial t} = -\frac{1}{m}\frac{\partial}{\partial x}\int_0^t \mathrm{d}t''\langle f(t'')\delta(\boldsymbol{u}-\boldsymbol{u}(t))\rangle - \frac{1}{m}\frac{\partial}{\partial v}\langle f(t)\delta(\boldsymbol{u}-\boldsymbol{u}(t))\rangle - v_0\frac{\partial p(x,v,t)}{\partial x}. \tag{11}$$

Since $\delta(\boldsymbol{u}-\boldsymbol{u}(t))$ is a functional of the Gaussian random noise at times t' prior to t, the averages appearing in (9) and (11) are given by Novikov's formula [10]

$$\langle f(t'')\delta(\boldsymbol{u}-\boldsymbol{u}(t))\rangle = \int_0^t \mathrm{d}t'\,\langle f(t'')f(t')\rangle\left\langle\frac{\delta[\delta(\boldsymbol{u}-\boldsymbol{u}(t))]}{\delta f(t')}\right\rangle \tag{12}$$

where the functional derivative obtained from the explicit solution of (7*a*, *b*) is

$$\frac{\delta[\delta(\boldsymbol{u}-\boldsymbol{u}(t))]}{\delta f(t')} = -\frac{1}{m}\left[(t-t')\frac{\partial}{\partial x}+\frac{\partial}{\partial v}\right]\delta(\boldsymbol{u}-\boldsymbol{u}(t)). \tag{13}$$

By substituting (12) and (13) into (9) and specializing to OU noise (5) one thus obtains the FPLE studied in I, namely

$$\frac{\partial p(x,v,t)}{\partial t} = -v\frac{\partial p}{\partial x} - b(t)\frac{\partial^2 p}{\partial x\partial v} + a(t)\frac{\partial^2 p}{\partial v^2} \tag{14}$$

where

$$a(t) = \eta(1-\mathrm{e}^{-(t/\tau)}) \tag{15a}$$

$$b(t) = \eta[(t+\tau)\mathrm{e}^{-(t/\tau)}-\tau] \qquad \eta = \frac{f_0^2}{2m^2}. \tag{15b}$$

On the other hand, from (11) we obtain in a similar fashion, for arbitrary Gaussian noise,

$$m^2\frac{\partial p(x,v,t)}{\partial t} = -v_0\frac{\partial p}{\partial x} + \int_0^t \mathrm{d}t'\int_0^t \mathrm{d}t''\langle f(t'')f(t')\rangle\left[(t-t')\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x\partial v}\right]p + \int_0^t \mathrm{d}t'\langle f(t)f(t')\rangle\left[(t-t')\frac{\partial^2}{\partial x\partial v}+\frac{\partial^2}{\partial v^2}\right]p. \tag{16}$$

In the special case of an OU correlation (5) for the noise this equation reduces to

$$\frac{\partial p(x,v,t)}{\partial t} = -v_0\frac{\partial p}{\partial x} + d(t)\frac{\partial^2 p}{\partial x^2} + c(t)\frac{\partial^2 p}{\partial x\partial v} + a(t)\frac{\partial^2 p}{\partial v^2} \tag{17}$$

where

$$c(t) = \eta[t(2-\mathrm{e}^{-(t/\tau)}) - \tau(1-\mathrm{e}^{-(t/\tau)})] \tag{18a}$$

$$d(t) = \eta[t^2 - t\tau(1-\mathrm{e}^{-(t/\tau)})]. \tag{18b}$$

Equation (17) coincides with equation (5.3) of Drory's paper [6]. Now, Drory has shown that the two FPLEs (14) and (17) have the same initial value or fundamental solution (i.e. the solution reducing to $\delta(x-x_0)\delta(v-v_0)$ at $t=0$) given by an exact expression which he has derived independently, i.e. without using a FPLE. This solution is

$$p(x,v,t) = \frac{1}{4\pi\sqrt{(\alpha\gamma-\beta^2)}}\exp\left\{-\frac{1}{4(\alpha\gamma-\beta^2)}[\gamma(x-x_0-v_0t)^2 - 2\beta(x-x_0-v_0t)(v-v_0)+\alpha(v-v_0)^2]\right\} \tag{19}$$

where the quantities

$$\alpha = \eta\left[\frac{t^3}{3} - \frac{1}{2}t^2\tau + \tau^3 - \tau^2(t+\tau)\mathrm{e}^{-(t/\tau)}\right] \tag{20a}$$

$$\beta = \frac{\eta}{2}[t^2 - t\tau(1-\mathrm{e}^{-(t/\tau)})] \tag{20b}$$

$$\gamma = \eta[t - \tau(1-\mathrm{e}^{-(t/\tau)})] \tag{20c}$$

are related to a, b, c, d as follows:

$$a(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \qquad b(t) = 2\left(\frac{\mathrm{d}\beta}{\mathrm{d}t} - t\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \qquad c(t) = 2\frac{\mathrm{d}\beta}{\mathrm{d}t} \qquad d(t) = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = 2\beta.$$

The expression (19) gives the distribution of the location and of the velocity of the particle about instantaneous mean values, $x_0 + v_0 t$ and v_0. These results show that the two FPLEs (14) and (17), which follow from a well defined procedure, are equally acceptable evolution equations for $p(x, v, t)$ since they have a common initial value solution with an obvious physical meaning. The physical relevance of the FLPE (17) is further emphasized by the following discussion.

A formal difference between the FPLEs (14) and (17) is revealed by integrating both sides over velocities to obtain an equation for the marginal distribution of displacements,

$$p(x, t) = \int_{-\infty}^{\infty} \mathrm{d}v\, p(x, v, t).$$

Assuming $p(x, v, t)$ and its derivatives to be well behaved at $x = \pm\infty$ and at $v = \pm\infty$, we thus obtain

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}v\, vp(x, v, t) \tag{21}$$

from (14), and

$$\frac{\partial p(x,t)}{\partial t} = -v_0\frac{\partial p}{\partial x} + d(t)\frac{\partial^2 p}{\partial x^2} \tag{22}$$

from (17), where $d(t)$ defined by (18*b*) is the time-dependent diffusion coefficient giving the mean-squared displacement [11]

$$\langle x^2(t)\rangle - (x_0 + v_0 t)^2 = 2\int_0^t d(t')\,\mathrm{d}t'. \tag{23}$$

It is seen that while (21) is just the continuity equation for the probability density, the diffusion-like (or Smoluchovski-like) equation (22) expresses probability conservation in terms of the contributions of drift- and diffusion-probability currents. However, the exact solution (19) readily permits us to reduce (21) to the form of the closed equation (22) for $p(x, t)$. This shows that (14) and (17) yield a unique closed equation for $p(x, t)$, the (initial value) solution of which, obtained by integrating (19) over velocities, is

$$p(x, t) = \frac{1}{2\sqrt{\pi\alpha(t)}}\exp\left(-\frac{1}{4\alpha(t)}(x - x_0 - v_0 t)^2\right). \tag{24}$$

The uniqueness of the FPLE (22) for $p(x, t)$ is related to the unique way of rewriting the Langevin equation (3) in the form of a first-order process, namely (10*a*). Indeed, the marginal distribution is defined in terms of the solution of (10*a*) by

$$p(x, t) = \langle\delta(x - x(t))\rangle. \tag{25}$$

The FPLE for $p(x,t)$ may be obtained by following the same steps as above, differentiating (25) with respect to t, inserting (10a) and, finally, performing the average of the expression for $\partial p(x,t)/\partial t$, using the form of (12) and (13) when only x is present as a variable. This yields

$$\frac{\partial p(x,t)}{\partial t} = -v_0 \frac{\partial p}{\partial x} + \frac{1}{m^2}\int_0^t \mathrm{d}t'\,(t-t')\int_0^t \mathrm{d}t''\,\langle f(t'')f(t')\rangle \frac{\partial^2 p}{\partial x^2}$$

for Gaussian noise with an arbitrary correlation $\langle f(t)f(t')\rangle \equiv \gamma(t,t')$. In the special case of OU noise (5) this equation reduces to the generalized diffusion equation (22).

Finally, we observe [6] that while the initial value—or so called fundamental solutions of (14) and (17) coincide, the mixed initial value–boundary value solutions [8] of these equations, when additional boundary conditions are imposed at a time t, are expected to be different (see, section 1). The diffusion-like equation (22) has been applied previously to the study of first passage times and to the related problem of survival probability in the presence of fixed traps [5], using the first passage condition for first-order stochastic processes [5, 12].

3. The Brownian motion process

As in the inertial case, the Langevin equation (1) for Brownian motion may be written in two different ways in the form of first-order equations, namely

$$\dot{x} = v \tag{26a}$$

$$\dot{v} = -\gamma v + \frac{1}{m} f(t) \tag{26b}$$

and

$$\dot{x} = v_0 \mathrm{e}^{-\gamma t} + \frac{1}{m}\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\gamma(t-t')} f(t') \tag{27a}$$

$$\dot{v} = -\gamma v + \frac{1}{m} f(t) \tag{27b}$$

where the right-hand side of (27a) is the general solution of (27b) in terms of an initial velocity v_0.

The FPLE describing the coupled second-order process (26a, b) is readily found by inserting these expressions in (8) and using the Novikov identity (12), with $\langle f(t)f(t')\rangle$ given by (2) and

$$\frac{\delta[\delta(\boldsymbol{u}-\boldsymbol{u}(t))]}{\delta f(t')} = -\frac{1}{m}\mathrm{e}^{-\gamma(t-t')}\left[(t-t')\frac{\partial}{\partial x} + \frac{\partial}{\partial v}\right]\delta(\boldsymbol{u}-\boldsymbol{u}(t)). \tag{28}$$

This yields ($\eta = f_0^2/2m^2$)

$$\frac{\partial p(x,v,t)}{\partial t} = -v\frac{\partial p}{\partial x} + \gamma\frac{\partial}{\partial v}(vp) + \eta\frac{\partial^2 p}{\partial v^2} \tag{29}$$

which is nothing but the familiar FPE, also called the free-particle Kramers equation, for the distribution $p(x,v,t)$ of Brownian motion (with the usual identification $\eta = \gamma kT/m$ resulting from the energy equipartition law).

On the other hand, by substituting (27a, b) in (8) we get

$$\frac{\partial p(x,v,t)}{\partial t} = -u_0(t)\frac{\partial p}{\partial x} + \gamma\frac{\partial}{\partial v}(vp)$$

$$-\frac{\partial}{\partial x}\int_0^t \mathrm{d}t''\,\mathrm{e}^{-\gamma(t-t'')}\langle f(t'')\delta(\boldsymbol{u}-\boldsymbol{u}(t))\rangle - \frac{\partial}{\partial v}\langle f(t)\delta(\boldsymbol{u}-\boldsymbol{u}(t))\rangle \tag{30}$$

where

$$u_0(t) = v_0\mathrm{e}^{-\gamma t} \tag{31}$$

denotes the instantaneous systematic velocity which is progressively damped out from its initial value, due to the frictional force. Using the Novikov formula (12) for the averages over the white noise in (30), where we now have (with $\theta(x)$ the step function)

$$\frac{\delta[\delta(\boldsymbol{u}-\boldsymbol{u}(t))]}{\delta f(t')} = \frac{1}{m}\left[\frac{1}{\gamma}(\mathrm{e}^{-\gamma(t-t')}-1)\frac{\partial}{\partial x} - \mathrm{e}^{-\gamma(t-t')}\theta(t-t')\frac{\partial}{\partial v}\right]\delta(\boldsymbol{u}-\boldsymbol{u}(t)) \tag{32}$$

we finally obtain the new FPLE

$$\frac{\partial p(x,v,t)}{\partial t} = -u_0(t)\frac{\partial p}{\partial x} + \gamma\frac{\partial}{\partial v}(vp) + \eta\frac{\partial^2 p}{\partial v^2} - r(t)\frac{\partial^2 p}{\partial x^2} + 2q(t)\frac{\partial^2 p}{\partial x\partial v} \tag{33}$$

where

$$q(t) = \frac{\eta}{2\gamma}(1-\mathrm{e}^{-2\gamma t}) \tag{34a}$$

$$r(t) = -\frac{\eta}{\gamma^2}(1-\mathrm{e}^{-\gamma t})^2. \tag{34b}$$

The next step of our discussion is to find the explicit initial value solution of the Kramers equation (29) and to show that this solution is also the initial value solution of our new FPLE (33). To this end, we define the double Fourier transform, $\tilde{p}(\xi,\nu,t)$, of the joint distribution $p(x,v,t)$:

$$\tilde{p}(\xi,\nu,t) = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{e}^{-\mathrm{i}\xi x-\mathrm{i}\nu v}p(x,v,t) \tag{35}$$

and by Fourier transforming (29) we get

$$\frac{\partial\tilde{p}(\xi,\nu,t)}{\partial t} = (\xi-\gamma\nu)\frac{\partial\tilde{p}}{\partial\nu} - \eta\nu^2\tilde{p}. \tag{36}$$

By expressing the solution of (36) in the form $\tilde{p}(\xi,\nu,t) = \tilde{p}_0(\xi,\nu)\tilde{q}(\xi,\nu,t)$, where $\tilde{p}_0(\xi,\nu)$ is the stationary solution given by

$$\tilde{p}_0(\xi,\nu) = \theta^{-(\eta/\gamma^3)\xi^2}\mathrm{e}^{-(\eta/2\gamma^3)\theta(\theta+4\xi)} \qquad \theta = \gamma\nu-\xi \tag{37}$$

one finds that $\tilde{q}$ obeys the homogeneous first-order equation

$$\frac{\partial\tilde{q}}{\partial t} = -(\gamma\nu-\xi)\frac{\partial\tilde{q}}{\partial\nu} \tag{38}$$

whose solution is

$$\tilde{q}(\xi,\nu,t) = \phi[(\gamma\nu-\xi)\mathrm{e}^{-\gamma t}]$$

where $\phi(z)$ is an arbitrary function of z. From the initial condition $p(x,v,0) = \delta(x-x_0)\delta(v-v_0)$, i.e. $\tilde{p}(\xi,\nu,0) = \exp(-\mathrm{i}\xi x_0 - \mathrm{i}\nu v_0)$ one then obtains $\phi(\gamma\nu-\xi) = [\tilde{p}_0(\xi,\nu)]^{-1}\exp(-\mathrm{i}\xi x_0-\mathrm{i}\nu v_0)$. This finally leads to an expression for $\tilde{p}(\xi,\nu,t)$ which may be reduced to the form

$$\tilde{p}(\xi,\nu,t) = \mathrm{e}^{-\mathrm{i}u_0\nu-\mathrm{i}w_0\xi-q\nu^2+r\xi\nu-s\xi^2} \tag{39}$$

where u_0, q and r are given by (31) and (34a, b), and

$$s(t) = \frac{\eta t}{\gamma^2} - \frac{\eta}{2\gamma^3}(1 - \mathrm{e}^{-\gamma t})(3 - \mathrm{e}^{-\gamma t}) \tag{40}$$

$$w_0(t) = x_0 + \frac{v_0}{\gamma}(1 - \mathrm{e}^{-\gamma t}). \tag{41}$$

Here $w_0(t)$ is the instantaneous systematic displacement which saturates at the value $x_0 + v_0/\gamma$ after time intervals long compared to the frictional relaxation time γ^{-1}. Finally, the exact initial value solution of (29) given by the inverse transform of (39) is

$$p(x, v, t) = \frac{1}{2\pi\sqrt{4qs - r^2}} \times \exp\left\{-\frac{1}{4qs - r^2}[q(x - w_0)^2 + r(x - w_0)(v - u_0) + s(v - u_0)^2]\right\}. \tag{42}$$

Now, it is not difficult to verify that (42) is also the exact initial value solution of the new FPLE (33). This verification is simplified by noting the following interrelations between the functions q, r and s

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \eta - 2\gamma q \tag{43a}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\gamma r - 2q \tag{43b}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -r \tag{43c}$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -2\gamma\Gamma + 4\eta s \qquad \Gamma = 4qs - r^2. \tag{43d}$$

We have thus shown that the Kramers equation (29) and the new FPLE (33), obtained as a direct consequence of the Langevin equations (1) in the form (27a, b), describe the same physics via their exact initial value solution (42). In particular, it follows that (33) fully incorporates the Markovian character of the Brownian motion process for large γ, via the explicit solution (42).

As in the case of the inertial process considered in section 2, we may obtain equations for the marginal distribution of displacements, $p(x, t)$, by integrating (29) and (33) over velocities. This yields, respectively, the continuity equation analogous to (21) and the closed diffusion-like equation

$$\frac{\partial p(x, t)}{\partial t} = -u_0(t)\frac{\partial p}{\partial x} - r(t)\frac{\partial^2 p}{\partial x^2} \tag{44}$$

with a generalized diffusion coefficient $D(t) = -r(t)$, which defines the mean-squared displacement

$$\langle x^2(t)\rangle - w_0^2(t) = -2\int_0^t r(t')\,\mathrm{d}t' = \frac{2\eta}{\gamma^2}\left[t - \frac{1}{2\gamma}(3 - 4\mathrm{e}^{-\gamma t} + \mathrm{e}^{-2\gamma t})\right]. \tag{45}$$

The correctness of this result is readily confirmed by direct evaluation of the white-noise average of $x^2(t)$ obtained from (27a). Also, with the help of (42), the equivalence of the continuity equation of the form (21) and of the diffusion-like equation (44) is readily demonstrated. The exact solution of (44) obtained by integrating (42) over velocities is

$$p(x, t) = \frac{1}{2\sqrt{\pi s(t)}}\exp\left[-\frac{1}{4s(t)}(x - w_0(t))^2\right]. \tag{46}$$

Again, the diffusion-like equation (44) may be derived directly from the definition (25) of $p(x,t)$ in terms of the solution $x(t)$ of the first-order Langevin equation (27*a*). The uniqueness of (44) is thus related to the unique way of expressing the Langevin equation (1) in the form of a closed first-order process for the particle's displacements.

As for the inertial case, we remark that, while the initial value solutions of the Kramers equation (29) and the FPLE (33) coincide, their mixed initial value–boundary value solutions, when boundary conditions at a finite t are added to the initial values, are generally expected to be different. Again, the generalized diffusion equation (44) allows one to study first passage times (as well as survival probabilities), using the familiar first passage condition [12] for first-order processes such as (27*a*).

4. Concluding remarks

Conventional treatments of stochastic processes [7, 9, 12] introduce a clear distinction between Markovian and non-Markovian processes. This is important not only because of the relative simplicity of the Markovian processes but also because these processes are of fundamental importance in applications, for example, in physics and chemistry. Markovian processes are characterized by the absence of memory; given the value obtained for a random variable at a time t, its evolution at later times depends only on the value at the time t, being independent of its value at times prior to t. For a Gaussian stochastic process, $y(t)$, the Markovian or non-Markovian character may be directly inferred from the solution of the Langevin equation, which determines the form of the correlation coefficient $\rho(t,t')$ defined by

$$\rho(t,t') = \frac{\langle y(t)y(t')\rangle}{(\langle y^2(t)\rangle\langle y^2(t')\rangle)^{1/2}} \tag{47}$$

where the angular brackets indicate averages over the random noise. Indeed, a necessary and sufficient condition for a Gaussian process to be Markovian is that $\rho(t,t')$ verifies the relation [14]

$$\rho(t,t') = \rho(t,t'')\rho(t'',t') \qquad t > t'' > t'. \tag{48}$$

Now, in the Brownian motion case it follows from (26*a*, *b*) that, for large $\gamma\,(\gamma t \gg 1)$,

$$v(t) \simeq \frac{1}{\gamma m} f(t) \qquad x(t) \simeq x_0 + \frac{1}{\gamma m}\int_0^t \mathrm{d}t'\, f(t'). \tag{49}$$

From (47) and (48) and (2) it then follows that both $v(t)$ and $x(t)$ are Markovian processes for $\gamma t \gg 1$. (Note that by using the explicit solution (27*a*) for the velocity one may verify that the latter is actually Markovian for all time scales.) On the other hand, one readily finds that the inertial process (10*a*), namely

$$x(t) = x_0 + v_0 t + \frac{1}{m}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, f(t'')$$

is non-Markovian (i.e. (48) is violated) for white noise as well as for OU noise (the OU noise itself is, however, Markovian, as is well known).

It follows from the above discussion that the FPLEs obtained directly from the Langevin equations fully incorporate the information about the Markovian or the non-Markovian nature of the considered processes. In the Brownian motion case a direct confirmation of this is provided by the fact that one of the FPLEs is the free-particle Kramers equation (29) obtained from the Fokker–Planck formalism for Markov processes. Furthermore, the other FPLE (33) has the same initial value solution as the Kramers equation and, is, therefore, also compatible with the Markovian nature of Brownian motion. We recall that, in contrast to the treatments of sections 3 and 4, the Fokker–Planck formalism is not primarily based on Langevin equations (which are used only to find the actual values of the coefficients in the FPE), but rather on the use of the Markovian property embodied in the Chapman–Kolmogorov equation (or its master equation version). On the other hand, we recall that while the discussion of the preceding paragraph emphasizes the distinction between the inertial- and the Brownian-motion processes from the point of view of the Markovian and/or the non-Markovian nature of the individual variables $x(t)$ and $v(t)$, this information does not fully characterize the two-variable process, as is well known [7, 14]. Thus for Brownian motion, as well as for inertial motion with $\tau \to 0$, the process $[x(t), v(t)]$ is always Markovian.

It is clear that the treatments of sections 2 and 3 leading to new FPLEs for the inertial- and Brownian-motion processes may be generalized in the case of an nth-order stochastic process $u(t)$ defined by a linear Langevin equation of nth-order. Such an equation may be rewritten either in the form of n coupled linear first-order equations for n stochastic variables $u_1 = u(t)$, $u_2 = \dot{u}(t)$, $u_3 = \ddot{u}(t), \ldots, u_n = \mathrm{d}^{n-1}u(t)/\mathrm{d}t^{n-1}$, or as a collection of n decoupled first-order processes expressed in terms of multiple time integrals of the noise $f(t)$. Alternatively the nth-order Langevin equation may also be cast into mixed forms involving coupled first-order equations for subsets of the n variables $u_1, u_2, \ldots, u_n$, together with Langevin equations for independent effective first-order processes for the remaining variables. To each different expression of the nth-order Langevin equation in the form of first-order differential equations there corresponds an FPLE of a particular form. As in the cases of second-order processes of sections 2 and 3, these different FPLEs are expected to have a common initial value solution for the joint distribution of the n random variables $u_1, u_2, \ldots, u_n$.

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